

Closed constraint algebras and path integrals for loop group actions

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Abstract

In this note we study systems with a closed algebra of second class constraints. We describe a construction of the reduced theory that resembles the conventional treatment of first class constraints. It suggests, in particular, to compute the symplectic form on the reduced space by a fiber integral of the symplectic form on the original space. This approach is then applied to a class of systems with loop group symmetry. The chiral anomaly of the loop group action spoils the first class character of the constraints but not their closure. Proceeding along the general lines described above, we obtain a 2-form from a fiber (path)integral. This form is not closed as a relict of the anomaly. Examples of such reduced spaces are provided by D-branes on group manifolds with WZW action.

1 Introduction

According to Dirac's classification, constraints in Hamiltonian mechanics split into first-class and second-class. The theory of the first-class constraints is well developed because it is a major tool in gauge theories. Second-class constraints naturally arise in gauge theories

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with anomalies: quantum corrections may cause first-class constraints of the classical system to become second-class [7].

We reconsider Dirac's approach to second-class constraints and give a new realization of the reduced phase space which is more in line with the reduction procedure for the first-class constraints. In this framework the Liouville form on the reduced phase space can be obtained by fiber integration from the Liouville form on the original phase space of the system.¹ The procedure can suffer from possible global difficulties similar to the Gribov problems one often encounters in the context of ordinary (anomaly-free) gauge theories.

Our main interest is to apply this formalism to loop group actions on symplectic manifolds. If the Poisson bracket of the symmetry generators contains a Schwinger term (for instance, this is the case in the WZW model), the constraints become second-class. Such a situation was considered in the mathematical literature [1],[2]. We use a fiber integration procedure to derive the Liouville form on the reduced phase space. As a new manifestation of the anomaly it turns out that this form is not closed (see also [2])!

'Anomalous' reduced spaces of this type naturally arise in the theory of D-branes on group manifolds [4]. It is an interesting problem to develop a consistent quantization theory for such spaces. Because of the connection between deformation quantization and open strings (see e.g. [10, 6, 11]) one expects valuable insights from open string theory. For the case of D-branes on group manifolds this was analysed in [3].

2 Dirac brackets from fiber integration

The aim of the present section is to reformulate the symplectic reduction for a system of constraints which form a closed algebra. To begin with we shall briefly recall the standard theory of reduction. This is used as a starting point for presenting an alternative formulation, applicable to closed constraint algebras under certain additional conditions. For the case of a purely second-class constraint system the approach suggests to construct the symplectic two-form and its associated Liouville form through a fiber integral. The material within this section serves as a toy model for the discussion of infinite dimensional phase spaces with an anomalous loop group action in Section 3 and 4.

Let us denote the original, unconstrained phase space by N and let x^i , $i = 1, \dots, 2n$, be local coordinates. Their Poisson bracket is denoted by $P^{ij} = \{x^i, x^j\}$ and its associated symplectic form by

$$\Omega = \frac{1}{2} \Omega_{ij} dx^i dx^j \quad .$$

$\Phi^\alpha = \Phi^\alpha(x)$, $\alpha = 1, \dots, 2m$, are constraints in this phase space. According to our assumptions they form a closed algebra, i.e.

$$\{\Phi_\alpha, \Phi_\beta\} = \Pi_{\alpha\beta}(\Phi) \tag{1}$$

¹We use the term 'Liouville form' for the exponential of a symplectic form

with a matrix Π that is a function of the constraints Φ_α only. For simplicity we also assume that the constraints are independent from one another (*irreducible* constraints) and that they are all *regular* so that they may be used as local coordinates in phase space, at least in a neighborhood of the constraint surface $\Phi(x) = 0$. Details and further results used in the text below can be found, e.g., in [9].

The standard procedure of symplectic reduction proceeds as follows: First the symplectic form Ω is pulled back to the constraint surface, which we denote by N_0 . The resulting two-form Ω_0 on N_0 is degenerate in general. Its kernel is surface-forming, however, and the quotient of N_0 with respect to the orbits (“gauge orbits”) is the reduced phase space M . By construction, the induced two-form ω on M is nondegenerate.

In the case of mere second-class constraints, i.e. $\det \Pi \neq 0$ on N_0 in our context, the last step does not arise, since Ω_0 is nondegenerate already. The Poisson bracket associated with Ω_0 may be obtained directly from the Poisson bracket on N by the following prescription

$$\{f, g\}_D := \{f, g\} - \{f, \Phi^\alpha\}(\Pi^{-1})_{\alpha\beta}\{\Phi^\beta, g\} \ ,$$

which is defined at least in some neighborhood of the constraint surface $N_0 \subset N$. As a bivector-field this bracket, known as the Dirac bracket, is *tangential* to the constraint surface $\Phi(x) = 0$ (in contrast to the original Poisson bracket). Hence, it has a push-forward to N_0 , and this coincides with the inverse of Ω_0 .

Due to the closedness of our constraint algebra (1), the Hamiltonian vector fields

$$v^\alpha := \{\Phi^\alpha, \cdot\} \equiv P^{ij} \frac{\partial \Phi^\alpha}{\partial x_i} \frac{\partial}{\partial x_j}$$

are surface-forming everywhere in N .² In fact, one can easily show that

$$[v^\alpha, v^\beta] = \frac{\partial \Pi^{\alpha\beta}}{\partial \Phi^\gamma} v^\gamma \ .$$

Thus the v^α s generate orbits in *all* of N and this is true even if the set (Φ^α) contains second-class constraints.

The last observation opens new possibilities for performing the symplectic reduction, viewing the reduced phase space M as an appropriate orbit space. The difference to the standard Dirac procedure outlined above lies primarily in the treatment of the second-class constraints, which are dealt with very analogously to first-class constraints in their standard reduction. Consequently, our main focus in the remainder of this section will be on second-class constraints. We will briefly comment on the extension to more general cases with the simultaneous presence of first- and second-class constraints towards the end of the section.

²For this the assumption on the closure of the constraint algebra is necessary as one can see e.g. from the fact that even a set of first-class constraints is surface-forming only along the constraint surface, in general.

For the case of pure second-class constraints, the matrix (1) is nondegenerate on the constraint surface, i.e. at $\Phi = 0$. In what follows we will strengthen this requirement by assuming that $\det \Pi(\Phi) \neq 0$ not only at the value zero, but for all values of Φ adopted. This permits us to regard the image C of the constraint map $\phi : N \rightarrow C \subset \mathbb{R}^{2m}$, $x \mapsto \Phi^\alpha(x)$, as a symplectic manifold; indeed $C \equiv \text{Im}\phi$ is endowed naturally with the symplectic form

$$\varpi = \frac{1}{2}(\Pi^{-1})_{\alpha\beta} d\Phi^\alpha d\Phi^\beta . \quad (2)$$

By construction, the map ϕ is Poisson.

In contrast to the standard approaches in which the reduced phase space M is regarded as a restriction of the original phase space N to the constraint surface $\Phi(x) = 0$, we propose to view M as the *space of orbits* generated by the second-class constraints Φ^α . Since the constraints are second-class, their Hamiltonian vector fields v^α are nowhere tangential to the constraint surface. Thus, at least locally, any point of the constraint surface, i.e. of the reduced phase space M , corresponds to an orbit (namely the one that is generated by the v^α s through the point in question).

Before we follow this general idea, we shall pause for a moment and comment the possible global difficulties. The full equivalence between the reduced phase space M and the orbit space requires any orbit to intersect the constraint surface once and only once. Despite the fact that we required Π to be nondegenerate everywhere, the orbits do not necessarily have this property in general. The situation we meet here is similar to the one of choosing gauge conditions for a set of first-class constraints, with $\det \Pi$ playing the role of the Faddeev–Popov determinant. Note that the combined system of first-class constraints and gauge-conditions forms a set of second-class constraints. It is known that even for a nonvanishing Faddeev–Popov determinant (along the intersection of the constraint surface with the gauge conditions) the chosen gauge conditions may show global deficiencies, in which case they are referred to as having a Gribov problem [8].

By analogy, we call the orbits generated by the v^α s to have a *Gribov problem*, if they do not intersect the constraint surface precisely once. To conclude these remarks, let us illustrate such problems through the following simple example where we take $N = T^*\mathbb{R} \setminus (0, 0)$ with standard symplectic form $\Omega = dq \wedge dp$. Now let us choose the constraints

$$\Phi^1 := \frac{(q^2 - p^2)}{2\sqrt{q^2 + p^2}} - \frac{1}{2} \quad \text{and} \quad \Phi^2 := \frac{qp}{2\sqrt{q^2 + p^2}} .$$

Their Poisson bracket is given by $\Pi^{12} = 1$ and one can easily establish that there is just *one* orbit in $T^*\mathbb{R} \setminus (0, 0)$. On the other hand, $T^*\mathbb{R} \setminus (0, 0)$ contains *two* points of the reduced phase space: $(q, p) = (\pm 1, 0)$. In fact, the map from $N = T^*\mathbb{R} \setminus (0, 0)$ to C defines a two-fold covering (as one may see most easily in polar coordinates). These constructions are easily extended to obtain examples with an arbitrary number of Gribov copies.

In the absence of a Gribov problem, however, the reduced phase space M may be fully identified with the space of orbits. Let π denote the projection from N to M along the

orbits. We then have the following proposition: The symplectic two-form ω on M satisfies

$$\pi^*\omega = \Omega - \phi^*\varpi . \quad (3)$$

The proof proceeds in several steps. First, we show that the form $(\Omega - \phi^*\varpi)$ descends to the space of leaves of the foliation. Indeed, it is horizontal,

$$\Omega(v^\alpha, \cdot) - \phi^*\varpi(u^\alpha, \cdot) = d\Phi_\alpha(x) - \phi^*d\Phi_\alpha = 0 . \quad (4)$$

Here u^α denotes the projection of the vector fields v^α to C (which is well-defined since the v^α s are tangential to the orbits): $u^\alpha = \{\Phi^\alpha, \cdot\}_C$, the index C being used to make clear that the bracket corresponds to (2). For later use we remark here that by assumption on the determinant of Π , the vector fields u^α — and thus also the vector fields v^α — are nonzero everywhere; correspondingly, the action generated by the constraints is free.

Equation (4) implies that, since the form in question is closed, it is also invariant with respect to the flows generated by the constraints. Hence, it is a pullback of some two-form on M , which we denote by ω .

Next, we show that ω coincides with the inverse of Dirac's bracket. For this purpose we consider two functions f and g on N which are constant on the leaves of the foliation. This implies that their Poisson brackets with the constraints vanish, yielding $\{f, g\}_D = \{f, g\}$. Denote the corresponding Hamiltonian vector fields by v_f and v_g . Note that one can use either the original Poisson bracket or the Dirac bracket to define them. We would like to show that

$$\omega(\pi_*v_f, \pi_*v_g) = \{f, g\}_D .$$

Indeed,

$$\{f, g\}_D = \{f, g\} = \Omega(v_f, v_g) = \omega(\pi_*v_f, \pi_*v_g) .$$

Here we have used that the vector fields v_f and v_g project to zero by ϕ . Thus we have fully established our formula (3) above.

In the presence of a Gribov problem with nonvanishing number of Gribov copies³ the map ϕ restricted to an orbit is not injective. For the following constructions we shall assume that the restriction of ϕ is a bijection. Note that this property is not guaranteed by the absence of a Gribov

problem, since ϕ may still fail to be surjective after restriction to an orbit. As an example we take N to be $T^*\mathbb{R}^2$ with the standard symplectic form and choose the constraints $\Phi^1 := \exp(q^2) [\exp(q^1) - 1]$, $\Phi^2 := \exp(-q^1 - q^2) p_1$, which again leads to $\Pi^{12} = 1$. Now, $C \equiv \text{Im}\phi = T^*\mathbb{R}$, but an orbit characterized by a fixed value of q^2 maps only to the parts of C with $\Phi^1 > -\exp(q^2)$.

If the map ϕ restricted to any orbit is surjective and there is no Gribov problem, the original phase space is a fiber bundle with typical fiber C and base manifold M . In this case there is an alternative way to express the relation between the form Ω on N and ω on

³One may also encounter situations in which some orbits do not intersect the constraint surface $\Phi(x) = 0$ at all; for an illustration just add the constant value 2 to the first constraint in the example below.

M : Let us consider the Liouville forms of mixed degree $L := \exp(\Omega)$ on N and $l := \exp(\omega)$ on M . The top degree components of L and l are the Liouville volume forms on N and M , respectively. We define the normalized fiber integral (or push forward map) π_* over the leaves of our foliation by the formula

$$\pi_* \alpha := \frac{1}{\text{Vol } C} \int_{\text{fiber}} \alpha . \quad (5)$$

Here α is a differential form on M and $\text{Vol } C$ is a (possibly infinite) symplectic volume of the constraint space C . If C is compact, π_* is just the ordinary push-forward map. Otherwise, the normalization factor $(\text{Vol } C)^{-1}$ is reminiscent of the infinite normalization constants in the definitions of path integrals. By applying the fiber integral (5) to the Liouville form L , we obtain

$$\begin{aligned} \pi_* L &= \frac{1}{\text{Vol } C} \int_{\text{fiber}} \exp(\Omega) = \frac{1}{\text{Vol } C} \exp(\omega) \int_{\text{fiber}} \exp(\Phi^* \varpi) \\ &= \frac{\int_C \exp(\varpi)}{\text{Vol } C} \exp(\omega) = l . \end{aligned} \quad (6)$$

In the next section we shall generalize equation (3) to an infinite dimensional context where Ω is given by a symplectic form on some space of fields. There will be one major difference in comparison to the considerations in the present section: the forms ω and ϖ will no longer be closed! The fiber integral (5) (which becomes a path integral) will then provide a prescription of how to define the Liouville form l for a nonclosed form ω .

Before turning to this, however, we briefly extend the above considerations to the general setting of a closed algebra of constraints, where there are both first- and second-class constraints. Note in this context that although one may always replace a set of constraints by an equivalent set of constraints where first- and second-class constraints are split [9], this splitting is achieved only on-shell (i.e. in a “weak sense”). On the full phase of the original theory, however, it may be impossible to find a splitting for which the second-class constraints do not generate first-class constraints upon Poisson commutation.

In the case of a closed constraint algebra containing first-class constraints, the matrix $\Pi(\Phi)$ is degenerate. Consequently, the manifold $C = \text{Im} \phi$ is no longer symplectic but only a Poisson manifold. Hence, C foliates into symplectic leaves. Let C_0 denote the symplectic leaf containing the origin $\Phi = 0$ and \tilde{N}_0 be the pre-image of C_0 , i.e. $\tilde{N}_0 = \phi^{-1}(C_0)$. (\tilde{N}_0 may be obtained equivalently through the action on N_0 of the flow generated by the constraints.) The reduced phase space may now be regarded as the space of orbits in \tilde{N}_0 , at least in the absence of a Gribov problem. A formula of the type (3) is true, if in the right-hand side Ω is the restriction of the symplectic form on the original space N to \tilde{N}_0 and ϖ is the symplectic form on C_0 . By means of such a formula one may, however, no more relate the Liouville forms on N and M such as in (6). The reason is that the fiber integration over Ω restricted to \tilde{N}_0 yields zero, since in the presence of first-class constraints this differential form has a kernel along the fibers.

3 Hamiltonian systems with loop group symmetry

Now we turn to the infinite dimensional situation of interest. Our phase space N is a field space with symplectic form Ω . By assumption, it has a Hamiltonian action of the loop group LG of some Lie group G , which we take to be compact, simple, and simply connected for simplicity. To an algebra element $\varepsilon(s) \in L\mathcal{G}$ we associate a Hamiltonian vector field

$$v_\varepsilon = \{J_\varepsilon, \cdot\} \quad (7)$$

on N , where

$$J_\varepsilon = \text{tr} \int_0^1 \varepsilon(s) J(s) ds \quad (8)$$

and $J(s)$ is a field giving rise to the moment map for the loop group action.

Using an orthonormal basis t^a in the Lie algebra \mathcal{G} , we can write the Poisson brackets of the components of $J(s)$ in the form

$$\{J^a(s), J^b(s')\} = k \delta^{ab} \delta'(s - s') + f_c^{ab} \delta(s - s') J^c(s) \quad (9)$$

Here k is a coefficient in front of the anomalous term in the bracket. We would like to use the currents $J^a(s)$ as constraints in our Hamiltonian system. If k vanishes, they are first-class constraints and can be treated by the standard procedure. Our main interest is to deal with the case of nonvanishing k . To simplify notations we will set $k = 1$ for the rest of the paper.

According to equation (9), the currents $J(s)$ form a closed algebra of both first- and second-class constraints. The zero modes of the currents $J_0^a := \int J^a(s) ds$ are first-class. All the remaining modes in a Fourier decomposition of $J(s)$ are second-class. The latter do not close among themselves since Poisson brackets of J_n with J_{-n} have J_0 -contributions. Hence, the Fourier modes J_n do not allow to split off a closed algebra of pure second class constraints. As we remarked above, such a splitting into closed first-class and closed second-class constraints need not even exist.

In the present case, however, we can split the constraints into first- and second-class. To see this, we return to the loop group LG , whose Lie algebra elements enter the Hamiltonians (8). LG may be written as a semidirect product of the group of based loops ΩG , formed by the loops with property $g(0) = e$, and the group G : Any $g(s) \in LG$ can be written uniquely as $g(s) = \tilde{g}(s) \hat{g}$ with $\tilde{g}(s) \in \Omega G$ and $\hat{g} \in G$. On the Lie algebra level this corresponds to the unique splitting of any $\varepsilon(s) \in L\mathcal{G}$ into the sum of a constant Lie algebra element $\varepsilon(0)$ and an $\tilde{\varepsilon}(s) \in \Omega\mathcal{G}$: $\varepsilon(s) = \varepsilon(0) + \tilde{\varepsilon}(s)$ with $\tilde{\varepsilon}(0) = 0$. Re-expressing the relations (9) in terms of the Hamiltonians (8), one finds

$$\{J_\varepsilon, J_\eta\} = \text{tr} \int_0^1 \varepsilon(s) \eta'(s) ds + J_{[\varepsilon, \eta]} \quad (10)$$

Since $\delta'(s - s')$ is an invertible operator on test-functions vanishing on the endpoints of the interval, these relations become second-class upon restriction to $\Omega\mathcal{G}$. Moreover, the algebra of this subclass of Hamiltonians is obviously closed now.

So, following the ideas of section 2, we should now be able to forget the first-class constraints and just restrict our attention to the subclass of second-class constraints so as to perform the pushforward integral (6) we are after. However, at this point we have to fight with the infinite dimensionality of the space of constraints and with the properties of an (appropriately defined) dual for the Lie algebra of the group ΩG . (Recall that the moment(um) map yields elements in the *dual* space of the Lie algebra of the group action in question, cf e.g. [12] for details.)

In this paper, we do not intend to go into the functional analytical details that would be necessary to fully and rigorously *extend* the approach of the previous section to the present infinite dimensional case (although this might yield interesting insights). Instead we will make use of a (mathematically rigorous) formula which is of the *form* of equation (3) with a (weakly) nondegenerate ϖ , which, however, is not closed and thus not symplectic.

For this purpose we return to the action of the group ΩG . As follows from equation (9), this group (or also LG) acts on the space of currents by standard gauge transformation,

$$J^g(s) = g^{-1} J g + g^{-1} \partial_s g \quad .$$

This action has no fixed points (here the restriction to ΩG becomes relevant!), similar to the flows of the vector fields u_α on C in Section 2. Hence, the action of ΩG on N is also free. Then, one can form the space of orbits, $M := N/\Omega G$ which replaces the space of leaves of the foliation of Section 2. The projection from N to M is denoted by π .

Similar to equation (3) we may decompose the symplectic form Ω on the original according to (cf. Theorem 8.3 in [1])

$$\Omega = \pi^* \omega + J^* \varpi \quad , \tag{11}$$

where ω is a two-form on M and ϖ lives on the space of currents (for a more precise definition of this space cf also [1]). The explicit formula for ϖ looks as follows. Denote by Ψ the solution of the equation

$$\partial_s \Psi \Psi^{-1} = J(s)$$

with the boundary condition $\Psi(0) = e$. In other words, $\Psi(s)$ is a path ordered exponential of $J(s)$ and $\Psi(1)$ is the holonomy map, which takes values in the group G . Obviously, this map descends to M and we shall denote the induced map by $\psi : M \rightarrow G$.

The form ϖ is given by

$$\varpi := \frac{1}{2} \text{tr} \int_0^1 (\Psi^{-1} d\Psi \partial_s (\Psi^{-1} d\Psi)) \quad . \tag{12}$$

As remarked above, it is not closed,

$$d\varpi = \frac{1}{6} \text{tr} (\Psi^{-1} (1) d\Psi(1))^3 \quad .$$

As a consequence of formula (11), the form ω is also not closed,

$$d\omega = -\frac{1}{6} \text{tr} (\psi^{-1} d\psi)^3 \quad .$$

Note, however, that the right-hand side of the last two formulas is proportional to the coefficient k in (9), which we have set to one thereafter. Thus, these forms become closed in the absence of the anomalous term in the current algebra. This observation will become relevant when interpreting the final result of the calculation in section 4.

Although ϖ is not symplectic, it comes very close to an inverse of the Poisson brackets (10) between the second-class constraints. By straightforward calculation one verifies the two relations

$$\iota(v_{\tilde{\varepsilon}})\varpi \equiv \varpi(v_{\tilde{\varepsilon}}, \cdot) = -dJ_{\tilde{\varepsilon}} , \quad \varpi(v_{\tilde{\varepsilon}}, v_{\tilde{\eta}}) = \{J_{\tilde{\varepsilon}}, J_{\tilde{\eta}}\} . \quad (13)$$

For these relations to hold it is essential that one restricts the Lie algebra elements to $\Omega\mathcal{G}$ (for the corrections appearing otherwise cf Proposition 8.1 in [1]). In the finite dimensional setting, equations of the form (13) for a complete set of Hamiltonian vector fields are already sufficient to ensure that ϖ is the sought-for symplectic form; the closedness would then follow automatically by validity of the Jacobi identity for the Poisson bracket.

In the present infinite dimensional setting, the form ϖ yielding relations of the form (13) is even not unique. Indeed, one can change the splitting (11) by an arbitrary 2-form β on the group G ,

$$\tilde{\omega} = \omega + \psi^*\beta , \quad \tilde{\varpi} = \varpi - \Psi^*\beta ,$$

without affecting the relations (13) where ϖ is replaced by $\tilde{\varpi}$. Note that because the 3-form $tr(\psi^{-1}d\psi)^3$ belongs to a nontrivial cohomology class on G , also the 2-form $\tilde{\omega}$ is not closed,

$$d\tilde{\omega} = d\omega + \psi^*d\beta \neq 0.$$

The phase space N is symplectic and carries the Liouville form $L = \exp(\Omega)$. The (formal) top degree part of L gives the measure of the Hamiltonian path integral. Inspired by equation (6), we would like to (formally) *define* the Liouville form l on M by the formula

$$l := \pi_* L = \frac{1}{\text{Vol}\Omega G} \int_{\Omega G} \exp \Omega \equiv \frac{\int_{\Omega G} \exp(\varpi)}{\text{Vol } \Omega G} \exp(\omega) , \quad (14)$$

where we made use of the definition (5) as well as of the relation (11). In the next section we compute the path integral

$$I(\psi) := \frac{1}{\text{Vol } \Omega G} \int_{\Omega G} \exp(\varpi) ,$$

where $\psi = \Psi(1)$ is an element of G . Note that the resulting integral will be a differential form of mixed degree rather than a function on G (or its pullback to M).

4 Evaluation of the path integral

We want to integrate $\exp(\varpi)$ over the group of based loops ΩG . Therefore we split the field $\Psi(s)$ in formula (12) into a product of an element $h \in \Omega G$ and an extra factor $\exp(\alpha s)$, i.e.

$$\Psi(s) = h(s) \exp(\alpha s) ,$$

where $h(s) \in G$ is a periodic G -valued function and $\alpha \in \mathcal{G}$ is sent to the group element $\Psi(1) = \exp(\alpha) \in G$ by the exponential mapping.

A short and elementary computation allows to reexpress the form ϖ in terms of the variables $h(s)$ and α . The result is,

$$\varpi = \frac{1}{2} \text{tr} \int_0^1 ds \left((h^{-1}dh) D_\alpha (h^{-1}dh) + 2h^{-1}dh d\alpha - d(e^{\alpha s}) e^{-\alpha s} d\alpha \right) .$$

D_α denotes the covariant derivative $D_\alpha = \partial_s - ad_\alpha$ where $ad_\alpha(\cdot) = [\alpha, \cdot]$. The last term in ϖ can be evaluated with the help of the following formula

$$\theta(s) := d(e^{\alpha s}) e^{-\alpha s} = \frac{1}{ad_\alpha} (1 - e^{sad_\alpha}) d\alpha . \quad (15)$$

Here, $1/ad_\alpha = (ad_\alpha)^{-1}$ is the inverse of the adjoint action ad_α with α . Note that the function $\frac{1}{x}(1 - e^{sx}) = \sum_{n \geq 1} s^n x^{n-1}/n!$ is regular even at $x = 0$ so that the right hand side of formula (15) is well-defined. To establish eq. (15) we differentiate the function $\theta(s)$ with respect to s to find,

$$\partial_s \theta(s) = d\alpha + [\alpha, \theta(s)] .$$

If the ansatz $\theta(s) = \exp(sad_\alpha) \vartheta(s)$ is inserted into the expression for $\partial_s \theta(s)$ we deduce

$$\partial_s \vartheta(s) = e^{-sad_\alpha} d\alpha .$$

This equation can easily be integrated to give the claimed formula for $\theta(s)$.

Formula (15) actually allows to perform the integral over s for the third term in ϖ . This results in

$$\varpi = \frac{1}{2} \text{tr} \int_0^1 ds (\phi D_\alpha \phi + 2\phi d\alpha) - \frac{1}{2} \text{tr} \left(d\alpha \frac{1}{(ad_\alpha)^2} (e^{ad_\alpha} - 1 - ad_\alpha) d\alpha \right) . \quad (16)$$

Again, the argument of the second trace is well-defined on the kernel of ad_α . In this expression for the form ϖ we also introduced the field $\phi(s) = h^{-1}(s)dh(s)$. By construction, $\phi(s)$ is a fermionic field subject to the constraint $\phi(0) = 0 = \phi(1)$. The integral over the exponential of the two-form ϖ is now reinterpreted as a fermionic “path integral” $\int \mathcal{D}\phi \exp(\varpi)$.

From the proof of eq. (15) above it is obvious that $D_\alpha \theta(s) = d\alpha$. Therefore one can rewrite the form ϖ also as

$$\varpi = \frac{1}{2} \text{tr} \int_0^1 ds (\phi + \theta) D_\alpha (\phi + \theta) .$$

This may lead one to conclude that the integration of $\exp \varpi$ over ϕ merely results in the Pfaffian of D_α . However, $(\phi + \theta)(s)$ does not vanish at $s = 1$ and a change of variables to $\phi + \theta$ is illegitimate.

We therefore proceed with integrating $\exp(\varpi)$ in the form of eq. (16). As the last term does not depend on ϕ (resp. h) and as, being a two-form, it commutes with the first two terms, we can split the exponential into two parts, the second one of which we may pull out of the integral, i.e. we shall write $\varpi = \varpi_1 - \varpi_2$ with

$$\varpi_1 = \frac{1}{2} \text{tr} \int_0^1 ds (\phi D_\alpha \phi + 2\phi d\alpha) , \quad (17)$$

$$\varpi_2 = \frac{1}{2} \text{tr} \left(d\alpha \frac{1}{(ad_\alpha)^2} (e^{ad_\alpha} - 1 - ad_\alpha) d\alpha \right) \quad (18)$$

and compute the Integral $I = \int \mathcal{D}\phi \exp(\varpi_1)$, leaving out the extra factor $\exp(-\varpi_2)$ for the moment.

The field $\phi(s)$ is a periodic fermionic field which admits the Fourier decomposition: $\phi(s) = \sum_n \phi_n \exp(2\pi i n s)$. In terms of the Fourier modes ϕ_n , the constraint $\phi(0) = 0$ becomes $\sum_n \phi_n = 0$. To turn the integral into a Gaussian one over *unrestricted* variables, we introduce a Lagrange multiplier λ . This leaves us with the computation of the following integral:

$$I = \int \prod_n d\phi_n d\lambda \exp \text{tr} \left(\frac{1}{2} \sum_m \phi_{-m} D_m \phi_m + \phi_0 d\alpha + \lambda \sum_m \phi_m \right) ,$$

where $D_n \equiv 2\pi i n - ad_\alpha$ and λ is a fermionic variable too. A product over the Lie algebra indices of ϕ_n and λ in the integration measure is understood, furthermore. Defining $J_n = \lambda + \delta_{n,0} d\alpha$, the second and third

term in the exponent may be combined into $\sum_m \phi_{-m} J_m$.

We would like to remark that the last reformulation of our integral involves the choice of some particular (anti)self-adjoint extension for the operator D_α : on its original domain of definition which consists of sections vanishing at both ends of the interval $[0, 1]$, iD_α is symmetric only, while on sections satisfying periodic boundary conditions it becomes self-adjoint.

The operator D_α is not invertible in the space of periodic sections. Its kernel is the “diagonal part” of the constant section. By “diagonal” we mean the subspace of the Lie algebra that commutes with α , thus being in the kernel of ad_α . We therefore integrate over ϕ_0^{diag} first. This produces a delta function $\delta(J_0^{diag}) \equiv \delta(\lambda^{diag} + (d\alpha)^{diag})$, which fixes the diagonal part of λ . On the remaining space the operator is invertible and we can perform the fermionic Gaussian integration, using

$$\int \mathcal{D}\psi \exp \left(\frac{1}{2} \psi_i \mathcal{O}_{ij} \psi_j + \psi_i J_i \right) = Pf(\mathcal{O}) \exp \left(\frac{1}{2} J_i \mathcal{O}_{ij}^{-1} J_j \right) . \quad (19)$$

Here, ψ and J have been taken fermionic, the operator \mathcal{O} was assumed to satisfy $\psi \mathcal{O} \psi = -(\mathcal{O} \psi) \psi$, and $Pf(\mathcal{O})$ denotes the Pfaffian of \mathcal{O} . We get

$$I = Pf(D_\alpha) \int d\tilde{\lambda} \exp \text{tr} \left(\frac{1}{2} \sum_n J_{-n} D_n^{-1} J_n \right) , \quad (20)$$

where the Pfaffian is taken over the space of periodic sections without kernel and $\tilde{\lambda}$ denotes the nondiagonal part of λ . Note that $J_0^{diag} \equiv 0$ so that the expression in the exponent is well-defined. Actually, since D_n becomes merely the number $2\pi i n$ on diagonal elements, all of the diagonal parts of J_n drop out due to $J_{-n} = J_n$ and the fermionic character of J_n . We indicate this again by means of tildes. Inserting the definition of J_n , eq. (20) becomes

$$I = Pf(D_\alpha) \int d\tilde{\lambda} \exp \operatorname{tr} \left[\frac{1}{2} \tilde{\lambda} \left(\sum_n D_n^{-1} \right) \tilde{\lambda} + \tilde{\lambda} (ad_\alpha)^{-1} \tilde{d\alpha} + \frac{1}{2} \tilde{d\alpha} (ad_\alpha)^{-1} \tilde{d\alpha} \right].$$

This is again a Gaussian integral for the variable $\tilde{\lambda}$ and we assume that α is sufficiently “generic” for $\sum_n D_n^{-1}$ to possess an inverse. We may again apply eq. (19) to obtain

$$I = Pf(D_\alpha) Pf\left(\sum_n D_n^{-1}\right) \exp \operatorname{tr} \left[-\frac{1}{2} \tilde{d\alpha} \frac{\left(\sum_n D_n^{-1}\right)^{-1}}{ad_\alpha^2} \tilde{d\alpha} + \frac{1}{2} \tilde{d\alpha} (ad_\alpha)^{-1} \tilde{d\alpha} \right],$$

where use of the ad-invariance of the trace (Killing metric) has been made.

This result for $I = \int \mathcal{D} \exp(\varpi_1)$ may now be combined with the expression for ϖ_2 in eq. (18) to yield

$$\int \mathcal{D}\phi \exp(\varpi) = Pf(D_\alpha) Pf\left(\sum_n D_n^{-1}\right) \exp \left[-\frac{1}{2} \operatorname{tr} \tilde{d\alpha} \frac{(\sum_n D_n^{-1})^{-1} + e^{ad_\alpha}}{ad_\alpha^2} \tilde{d\alpha} \right].$$

Here we made use of the fact that the diagonal parts of $d\alpha$ drop out in (18) and that $\operatorname{tr}(\tilde{d\alpha} f(ad_\alpha) \tilde{d\alpha})$ vanishes for any function f with $f(x) = f(-x)$. We are left with the computation of the operator $\sum_n D_n^{-1}$ and the two Pfaffians.

We start with $Pf(D_\alpha)$. Denote by $i\alpha_r$ the nonvanishing eigenvalues of ad_α , which are purely imaginary as G is taken compact. The index r runs over all roots in the Lie algebra of G ; with $r > 0$ ($r < 0$) labeling the positive (negative) roots, one has $\alpha_r = -\alpha_{-r}$, furthermore. In this notation one finds the following formal expression for the Pfaffian,

$$Pf(D_\alpha) = \left(\prod_{r>0} i\alpha_r \right) \prod_{n>0} \left((2\pi i n)^{\operatorname{rank} G} \prod_{r'} (2\pi i n + i\alpha_{r'}) \right).$$

Clearly this is not well-defined. However, integrating $\exp \varpi$ over all of ΩG we cannot expect to obtain a finite result as the volume of the “gauge group” ΩG is infinite. So we should divide (again formally) by this volume. The group of (based) loops is a group of even cohomology, $\varepsilon = \int_0^1 ds \operatorname{tr} (h^{-1} dh \partial_s h^{-1} dh)$ being the generator of $H^2(\Omega G)$. So, formally the Haar measure of ΩG is given by the infinite product of ε ’s multiplied by the Haar measure on G (since the zero mode drops out from ε). Using our previous notation

and Fourier decomposition, ε may be rewritten as $\text{tr} \sum_{n \neq 0} (2\pi i n) d\phi_{-n} d\phi_n$. Thus we are led to *define*:

$$Pf(D_\alpha)/\text{Vol} \Omega G := \left(\prod_{r>0} \alpha_r \right) \prod_{n>0} \prod_{r'} \left(1 + \frac{\alpha_{r'}}{2\pi n} \right) .$$

By means of $\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2} \right)$ we then obtain

$$Pf(D_\alpha)/\text{Vol} \Omega G = \prod_{r>0} \left(\sin\left(\frac{\alpha_r}{2}\right) \right) . \quad (21)$$

We remark that the square of this result agrees with the expression obtained for $\det D_\alpha$ obtained in [5] by means of zeta function regularization.

We now come to the operator $\sum_n D_n^{-1}$, $D_n \equiv 2\pi i n - ad_\alpha$, acting in that part of the Lie algebra that does not commute with α . Here we may use the simple formula

$$\sum_n \frac{1}{2\pi i n - x} = \frac{1}{2} \coth\left(\frac{x}{2}\right)$$

to conclude that

$$\sum_n D_n^{-1} = \frac{1}{2} \coth(ad_\alpha/2) \quad \text{and thus} \quad \left(\sum_n D_n^{-1} \right)^{-1} = 2 \tanh(ad_\alpha/2) .$$

Putting all this together, we arrive at the following result:

$$\frac{\int \mathcal{D}\phi \exp(\varpi)}{\text{Vol} \Omega G} = \prod_{r>0} \cos\left(\frac{\alpha_r}{2}\right) \exp \left[-\text{tr} d\alpha \frac{\sinh^3(ad_\alpha/2)}{ad_\alpha^2 \cosh(ad_\alpha/2)} d\alpha \right] .$$

Again, we have replaced $\widetilde{d\alpha}$ by $d\alpha$ as the extra contributions involving $(d\alpha)^{diag}$ cancel anyway. We can finally rewrite the two-form in the exponent in terms of the group element $\psi = \exp(\alpha)$. First, we remark that

$$\prod_{r>0} \cos\left(\frac{\alpha_r}{2}\right) = \det^{\frac{1}{2}} \left(\frac{1 + Ad_\psi}{2} \right) ,$$

where $\det^{\frac{1}{2}}$ denotes the unique positive square root of the matrix $(1 + Ad_\psi)/2$. Next, the formula (15) in Lemma 1 can be evaluated at $s = 1$ to give $d\psi\psi^{-1} = ad_\alpha^{-1}(1 - e^{ad_\alpha})d\alpha$. This may be inserted into our previous result for the integral and leads to

$$I(\psi) = \frac{\int \mathcal{D}\phi \exp(\varpi)}{\text{Vol} \Omega G} = \det^{\frac{1}{2}} \left(\frac{1 + Ad_\psi}{2} \right) \exp \frac{1}{4} \left[\text{tr} d\psi\psi^{-1} \frac{Ad_\psi - 1}{Ad_\psi + 1} d\psi\psi^{-1} \right] . \quad (22)$$

5 Results and Discussion

Combining equations (6) and (22), we obtain the expression

$$l = \det^{\frac{1}{2}} \left(\frac{1 + Ad_\psi}{2} \right) \exp \left(\omega + \frac{1}{4} \text{tr} d\psi \psi^{-1} \frac{Ad_\psi - 1}{Ad_\psi + 1} d\psi \psi^{-1} \right) . \quad (23)$$

for the Liouville form l on the orbit space M . The same expression was previously used in [2] (formula (21)). Our path integral consideration gives a natural derivation of equation (23), and shows its relation to the Liouville form L on the field space N .

Let us recall on this occasion that there was some freedom in our computation associated with the choice of an anti-self-adjoint extension for D_α . Instead of the periodic boundary conditions we introduced in the paragraph below eq. (18), we could have extended the antisymmetric operator D_α also to sections with different (only quasi-periodic) behaviour at the boundary. The final formula for $I(\psi)$ does depend on this choice of boundary conditions. It is expected, however, that the top degree part of the Liouville form l is insensitive to this freedom in the computation.

Recall that the space M arises as a result of reduction from the field space N with respect to second-class constraints. The residual first-class constraints J_0^a generate vector fields v_a on N which descend to M . According to [2], Proposition 4.1, the Liouville form l satisfies the following interesting equation,

$$\left(d + \frac{1}{24} f_{abc} \iota(v_a) \iota(v_b) \iota(v_c) \right) l = 0 . \quad (24)$$

Note that in the finite dimensional case of Section 2, $l = \exp(\omega)$ is a closed form. In the infinite dimensional situation we obtain an extra term $\frac{1}{24} f_{abc} \iota(v_a) \iota(v_b) \iota(v_c)$ on the left hand side of equation (24), which modifies the exterior differential and should be interpreted as yet another manifestation of the chiral anomaly. It is a very interesting open question to trace the nature of this anomaly back to properties of the path integral in Section 4.

Simple examples of spaces M are given by D-branes in the WZW model [4]. There, the reduced spaces are conjugacy classes in a group manifold, and the form ω is given by the formula (see equation (7) in [4]),

$$\omega = -\frac{1}{4} \text{tr} \left(d\psi \psi^{-1} \frac{Ad_\psi + 1}{Ad_\psi - 1} d\psi \psi^{-1} \right) .$$

Formula (23) shows that the form ω should be corrected by the extra term arising from the path integral to yield

$$\tilde{\omega} = -\text{tr} \left(d\psi \psi^{-1} \frac{1}{Ad_\psi - Ad_\psi^{-1}} d\psi \psi^{-1} \right) .$$

Note that in this case, the linear map $4^{-1}(Ad_\psi - 1)(Ad_\psi + 1)^{-1}$ representing the correction term in eq. (23) is inverse to the element $B = 4^{-1}(Ad_\psi + 1)(Ad_\psi - 1)^{-1}$ that appears in ω .

Hence, their difference $\tilde{\omega}$ is represented by $B - B^{-1}$. Surprisingly, the same combination shows up in the expression for the effective B -field derived in [11] in the analysis of D-branes on the flat background. It is another challenging question to understand why the formula of [11] applies to group manifolds and to establish the relation with the path integral of Section 3.

In this paper we did not touch the issue of quantization of the spaces N and M . While one can attempt to quantize N using the symplectic form Ω , it is not clear what it means to quantize M because the form ω is not closed. In the case of the D-branes in the WZW model one can use the link between string theory and noncommutative geometry to obtain an answer to this question [3]. The general case, however, remains an open problem.

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